

A Dehn surgery description of regular finite cyclic covering spaces of rational homology spheres

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Abstract

We provide related Dehn surgery descriptions for rational homology spheres and a class of their regular finite cyclic covering spaces. As an application, we use the surgery descriptions to relate the Casson invariants of the covering spaces to that of the base space. Finally, we show that this places restrictions on the number of finite and cyclic Dehn fillings of the knot complements in the covering spaces beyond those imposed by Culler-Gordon-Luecke-Shalen and Boyer-Zhang.

Keywords: Covering space; Dehn surgery; Casson invariant; Dehn filling

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1 Introduction

It is well-known that any closed, oriented, connected 3-manifold may be obtained by Dehn surgery on a link in S^3 . In recent years, 3-manifold theorists have exploited this fact, computing certain 3-manifold invariants by describing how the invariant changes under Dehn surgery. This has been an effective method for computing invariants of individual manifolds. However, until now, there was no known procedure for relating Dehn surgery descriptions of manifolds with those of their covering spaces. Therefore, although invariants for a manifold and a covering space of the manifold could each be computed using Dehn surgery formulas, no general statements could easily be made regarding the relationship between the invariants.

Let (X, \tilde{X}) be a 3-manifold pair, where X is a rational homology sphere and \tilde{X} is a regular finite cyclic covering space of X , say with $\pi_1(X)/\phi_*(\pi_1(\tilde{X})) = \mathbb{Z}/k\mathbb{Z}$, where $\phi : \tilde{X} \rightarrow X$ is the projection map. Since this group is abelian, we may factor the quotient map $\pi_1(X) \rightarrow \pi_1(X)/\phi_*(\pi_1(\tilde{X}))$ through the first homology group $H_1(X; \mathbb{Z})$.

Definition 1.1 We call the covering $\tilde{X} \rightarrow X$ **torsion-split** if there exists a homology decomposition $H_1(X; \mathbb{Z}) = \mathbb{Z}/kp\mathbb{Z} \oplus H$ (where possibly $H = 0$) satisfying:

- i) the decomposition is a decomposition of the torsion linking pairing on $H_1(X)$: i.e. if α is a generator of the $\mathbb{Z}/kp\mathbb{Z}$ -summand and $\beta_1, \beta_2, \dots, \beta_j$ is a torsion basis for H , then $\text{link}(\alpha, \alpha) = m/n$ for some m relatively prime to n and $\text{link}(\alpha, \beta_i) = 0$ for all i
- ii) any generator of the $\mathbb{Z}/kp\mathbb{Z}$ -summand maps to a generator in $\pi_1(X)/\phi_*(\pi_1(\tilde{X}))$ under the quotient map $H_1(X) \rightarrow \pi_1(X)/\phi_*(\pi_1(\tilde{X}))$
- iii) H maps to 0 in $\pi_1(X)/\phi_*(\pi_1(\tilde{X}))$.

In this paper, we provide a Dehn surgery description for torsion-split regular k -fold cyclic covering space pairs (X, \tilde{X}) with base space a rational homology sphere. Specifically, we prove the following:

Let K be a knot in S^3 , and let $L = (L_1, L_2, \dots, L_n)$ be a link in S^3 . Assume K bounds a Seifert surface Σ with the property that there exist Seifert surfaces $\Sigma_1, \Sigma_2, \dots, \Sigma_n$ for L_1, L_2, \dots, L_n disjoint from a neighborhood of Σ . Let p, q , and k be integers with $1 \leq q$, $1 \leq |p|$, and $k > 1$. Assume kp and q are relatively prime. Let M be the 3-manifold obtained by kp/q -Dehn surgery on K in S^3 followed by surgery on L with surgery coefficients $I = (i_1, i_2, \dots, i_n)$, where $i_j = \pm 1$ for each j .

Note that for each $j = 1, 2, \dots, n$, the component L_j of L has k disjoint lifts in the k -fold branched cyclic cover of S^3 branched along K , since (K, L_j) is a boundary link. Let \tilde{M} be the 3-manifold obtained by p/q -Dehn surgery on the lift of K to the k -fold branched cyclic cover of S^3 branched along K , followed by surgery on the k lifts of L with surgery coefficient i_j for every lift of L_j .

Theorem 1.2 \tilde{M} is a regular k -fold cyclic covering space of M .

Call (K, L, k, p, q, I) a *pairwise Dehn surgery description* for (M, \tilde{M}) . Then we also have

Theorem 1.3 Let (X, \tilde{X}) be a torsion-split regular k -fold cyclic covering space pair with base space X a rational homology sphere. Then (X, \tilde{X}) has a pairwise Dehn surgery description.

Thus, every torsion-split regular k -fold cyclic covering space pair over a rational homology sphere has a pairwise Dehn surgery description. This, then, may be used for computing 3-manifold invariants for the pair. This should be a necessary first step in drawing general conclusions about the relationships between invariants for X and those for \tilde{X} .

The paper is outlined as follows. In Section 2, we show that the manifolds generated by the pairwise Dehn surgery description (K, L, k, p, q, I) are a regular k -fold cyclic covering space pair. In Section 3, we show that all torsion-split regular k -fold cyclic covering space pairs over rational homology spheres arise in this way. In Section 4, as an application, we compute Casson invariants for the pair (X, \tilde{X}) .

In Section 5, we discuss the implications of our work for generating examples of finite and cyclic Dehn fillings of 3-manifolds with toral boundary. Specifically, let K be a knot homologous to 0 in a closed, connected, oriented 3-manifold Y , and let X be the result of p/q -Dehn surgery on K in Y . We show that if $\pi_1(X)$ is finite (resp. cyclic), then so too is the fundamental group of the 3-manifold obtained by r/q -Dehn surgery on the lift of K to the p/r -fold cyclic branched cover of Y branched along K , where r is any positive integer dividing p . Thus, one example of a finite or cyclic Dehn filling may generate many more such examples. Moreover, this restricts the possible number of finite and cyclic surgery slopes in the cyclic covering spaces of $Y - nbhd(K)$.

Finally, in the appendix, we demonstrate some relationships between certain Alexander polynomials which are required in Section 4.

We remark that the construction presented here may generalize to pairs (X, \tilde{X}) for which X is not a rational homology sphere. This theory, together with applications, will be the topic of a future paper.

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2 Generating covering space pairs

Retain all notation from Section 1. Further, denote by \overline{S}_K^3 the k -fold cyclic covering space of S^3 branched over K . Let N be the closed 3-manifold resulting from kp/q -Dehn surgery on K in S^3 , and let \tilde{N} denote the 3-manifold which is the result of p/q -Dehn surgery on the lift of K to \overline{S}_K^3 . Finally, if Y is any 3-manifold and K is a knot in Y , we abuse notation and denote by $Y - K$ the compact 3-manifold formed by removing an open tubular neighborhood of K .

Before proving Theorem 1.2, we prove the following:

Lemma 2.1 *\tilde{N} is a regular k -fold cyclic covering space of N .*

Proof of Lemma 2.1: Let \overline{K} denote the lift of K to \overline{S}_K^3 , and let \overline{m} denote its meridian. Let m denote the meridian of K . Let l denote the preferred longitude of K on the boundary of N .

$S^3 - K$, and let \bar{l} be a longitude in the boundary of $\bar{S}_K^3 - \bar{K}$ which lifts l . Then \tilde{N} is obtained by gluing a solid torus to $\bar{S}_K^3 - \bar{K}$, sending a meridian of the solid torus to the simple closed curve $\bar{m}^p \bar{l}^q$. Similarly, N is obtained by gluing a solid torus to $S^3 - K$, sending a meridian of the solid torus to $m^{kp} l^q$.

Let ϕ denote the regular k -fold cyclic covering map of the knot complements. We wish to show that ϕ extends to a regular k -fold cyclic (unbranched) covering map $\tilde{N} \rightarrow N$.

Note that ϕ restricts to a regular k -fold cyclic covering map from the boundary of $\bar{S}_K^3 - \bar{K}$ to $S^3 - K$. Thus, the map from the boundary of the solid torus in \tilde{N} to the boundary of the solid torus in N is a regular k -fold cyclic covering map.

Now a k -fold cyclic covering map from the boundary of one solid torus to the boundary of another solid torus extends to a regular k -fold cyclic covering map from the solid torus to the solid torus if and only if the meridian of the solid torus in the initial solid torus is taken to the meridian of the final solid torus. In our case, the meridian of the solid torus in \tilde{N} is $\bar{m}^p \bar{l}^q$, and the meridian of the solid torus in N is $m^{kp} l^q$. But ϕ takes \bar{m} to m^k and \bar{l} to l . Hence, $\phi(\bar{m}^p \bar{l}^q) = m^{kp} l^q$. It follows that ϕ takes a meridian of the solid torus in \tilde{N} to a meridian of the solid torus in N , and therefore ϕ extends to a covering map $\tilde{N} \rightarrow N$. \square

We now prove Theorem 1.2.

Proof of Theorem 1.2: We know that \tilde{N} is a regular k -fold cyclic covering space of N by Lemma 2.1. Let ϕ denote the covering map. We abuse notation and denote the image of L_j in N by L_j for $j = 1, 2, \dots, n$. Let \bar{L}_j denote the inverse image under ϕ of L_j for each j . Since (K, L_j) is a boundary link in S^3 for each j , we see that \bar{L}_j consists of k disjoint simple closed curves in \tilde{N} . Choose a Seifert surface Σ for K and Seifert surfaces Σ_j for L_j in S^3 such that $\Sigma \cap \Sigma_j = \emptyset$. Then the image of Σ_j in N is a Seifert surface for L_j in N , and it lifts to k Seifert surfaces $\bar{\Sigma}_{j,1}, \bar{\Sigma}_{j,2}, \dots, \bar{\Sigma}_{j,k}$ for the k lifts $\bar{L}_{j,1}, \bar{L}_{j,2}, \dots, \bar{L}_{j,k}$ of L_j in \bar{L}_j .

Recall that \bar{S}_K^3 may be explicitly constructed according to the following outline: let $Y^0 = \Sigma \times (-1, 1)$ be an open bicollar of Σ , and let $Y = Y^0 / (K \times (-1, 1) \sim K)$. Let Y^- be the manifold obtained by removing K from Y . Glue k copies of $S^3 - \Sigma$ together along k copies of Y^- , alternating copies of $S^3 - \Sigma$ with copies of Y^- . Finally, glue K back in to compactify. For a precise description of this construction, see [R, pp 128 - 131 and pp 297-298].

Now since (K, L_j) is a boundary link in S^3 , we see that the k lifts of Σ_j to \bar{S}_K^3 are contained in the k disjoint copies of $S^3 - Y$. It follows that these lifts of Σ_j are disjoint. Then clearly the Seifert surfaces $\bar{\Sigma}_{j,1}, \bar{\Sigma}_{j,2}, \dots, \bar{\Sigma}_{j,k}$ in \tilde{N} are also disjoint.

Now the Dehn surgery on L_j may be carried out in a neighborhood Z_j of Σ_j . Moreover, if we take Z_j to be sufficiently small, then the inverse image $\phi^{-1}(Z_j)$ consists of k disjoint copies $\bar{Z}_{j,i}$ of Z_j which are neighborhoods of the $\bar{\Sigma}_{j,i}$. Clearly the Dehn surgery on L_j in Z_j induces an identical Dehn surgery on $\bar{L}_{j,i}$ in $\bar{Z}_{j,i}$. Therefore every point of M has k distinct inverse

images in \tilde{M} , each with a neighborhood which is carried homeomorphically to a neighborhood of the point in M . Thus, \tilde{M} is a k -fold covering space of M .

Finally, note that for each j , the automorphism group $\mathbb{Z}/k\mathbb{Z}$ of $\phi : \tilde{N} \rightarrow N$ cyclically permutes the k disjoint copies of the Seifert surface for L_j in \tilde{N} . Clearly, then, the automorphism group of the covering space $\tilde{M} \rightarrow M$ is also $\mathbb{Z}/k\mathbb{Z}$, and the images of the k lifts of $\overline{L}_{j,i}$ after Dehn surgery are permuted by the automorphism group. Since the covering space $\tilde{N} \rightarrow N$ was regular, it follows that $\tilde{M} \rightarrow M$ is regular. \square

We remark that the covering $\tilde{M} \rightarrow M$ is clearly torsion-split by construction.

3 Completeness of the construction

In this section, we show that the construction described in Section 1 is complete in that it generates all torsion-split regular k -fold cyclic covering space pairs (X, \tilde{X}) over rational homology spheres. We prove:

Theorem 1.3 *Let (X, \tilde{X}) be a torsion-split regular k -fold cyclic covering space pair with base space X a rational homology sphere. Then (X, \tilde{X}) has a pairwise Dehn surgery description.*

We first prove the following lemma, which is a straightforward generalization of a lemma of S. Boyer and D. Lines [BL].

Lemma 3.1 *Let W be a rational homology sphere with $H_1(W; \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z} \oplus H$ for some finite abelian group H . Assume further that the homology decomposition arises as a decomposition of the torsion linking pairing on $H_1(W)$. Then there is a 3-manifold V with $H_1(V; \mathbb{Z}) = H$, a knot \mathcal{K} homologous to 0 in V , and an integer m such that W is the result of n/m -Dehn surgery on \mathcal{K} in V .*

Proof of Lemma 3.1: Let α be a generator of the $\mathbb{Z}/n\mathbb{Z}$ summand of $H_1(W; \mathbb{Z})$. Represent α by a curve C in W .

Note that the torsion subgroup of $H_1(W - C; \mathbb{Z})$ is just H . To see this, note that by assumption, $\text{link}(\alpha, \alpha) = t/n$ for some integer t relatively prime to n , where $\text{link}(-, -)$ is the torsion linking pairing on $H_1(W)$. Therefore C intersects the surface with boundary nC , and nC does not bound in $W - C$. On the other hand, there exist generators β_i of H such that $\text{link}(\alpha, \beta_i) = 0$. It follows that β_i has the same finite order in $H(W - C)$ as in $H(W)$.

Let $T(C)$ be a tubular neighborhood of C . Then $H_2(W, W - T(C); \mathbb{Z})$ is infinite cyclic with generator a meridional disk of $T(C)$. From the exact sequence

$$0 \rightarrow H_2(W, W - T(C); \mathbb{Z}) \rightarrow H_1(W - T(C); \mathbb{Z}) \rightarrow H_1(W; \mathbb{Z}) \rightarrow 0$$

it follows that $H_1(W - T(C); \mathbb{Z}) \cong \mathbb{Z} \oplus H$.

Let C' be a simple closed curve on $\partial T(C)$ generating the infinite cyclic summand of $H_1(W - T(C); \mathbb{Z})$. Attach a solid torus to $W - T(C)$ sending the meridian to C' ; call the resulting manifold V . Let \mathcal{K} denote the core of the surgery torus. Then $H_1(V; \mathbb{Z}) \cong H$, and W is the result of n/m -Dehn surgery on \mathcal{K} for some integer m . Moreover, \mathcal{K} is homologous to 0 in V , since all curves on $\partial T(C)$ represent 0 in H . \square

We now prove the theorem.

Proof of Theorem 1.3: Let $\tilde{X} \rightarrow X$ be a torsion-split regular k -fold cyclic covering space with X a rational homology sphere. Let $H_1(X; \mathbb{Z}) = \mathbb{Z}/kp\mathbb{Z} \oplus H$ be a homology decomposition for X obeying properties *i*) - *iii*) of Definition 1.1. We may apply Lemma 3.1 to find a 3-manifold V with $H_1(V; \mathbb{Z}) = H$, a knot \mathcal{K} homologous to 0 in V , and an integer q such that X is the result of kp/q -Dehn surgery on \mathcal{K} in V . Let Σ be a Seifert surface for \mathcal{K} in V .

It is well-known that there exists a link $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n)$ in V such that S^3 is the result of surgery on \mathcal{L} . Let $L = (L_1, L_2, \dots, L_n)$ be the image of \mathcal{L} in S^3 . Then V may be obtained from S^3 by surgeries on the components of L . Moreover, we may choose \mathcal{L} in such a way that the surgery coefficients for the components of L are all ± 1 .

Choose $\alpha_1, \alpha_2, \dots, \alpha_{2g}$ a collection of simple closed curves on Σ representing a homology basis for Σ . Isotope \mathcal{L} without changing any crossings of \mathcal{L} so that the linking number of \mathcal{L}_i and α_j is 0 for any $i = 1, 2, \dots, n$ and any $j = 1, 2, \dots, 2g$ and so that $\mathcal{L}_i \cap \Sigma = \emptyset$. We continue to denote by \mathcal{L} and L the images of \mathcal{L} and L under the isotopy. Let K be the image of \mathcal{K} in S^3 after surgery on \mathcal{L} . Abusing notation yet again, we denote by Σ and α_j the image of Σ and α_j in S^3 . Note that the linking number of L_i and α_j is 0 for any $i = 1, 2, \dots, n$ and any $j = 1, 2, \dots, 2g$ by Lemma A.5. Furthermore L_i does not meet Σ .

We require the following

Lemma 3.2 *Let C be a knot in S^3 , and let S be a Seifert surface for C . Let x_1, x_2, \dots, x_{2g} be a collection of curves on S representing a basis for $H_1(S; \mathbb{Z})$. Let D be a knot in S^3 such that $D \cap S = \emptyset$ and $lk(x_i, D) = 0$ for $i = 1, 2, \dots, 2g$. Then D bounds a Seifert surface disjoint from S .*

Proof of lemma: Let S' be a Seifert surface for D meeting S transversely. If $S' \cap S = \emptyset$, we are done. Otherwise, $S' \cap S$ is a collection y_1, y_2, \dots, y_m of oriented simple closed curves. Now the homology class represented by $y_1 + y_2 + \dots + y_m$ in $H_1(S; \mathbb{Z})$ must be 0. For if this class were non-zero, then there would be a curve x_i on S whose oriented intersection number with $y_1 + y_2 + \dots + y_m$ was non-zero. Then x_i would have non-zero intersection number with S' and hence would have non-zero linking number with D .

Now remove from S' the components of $S' - S$ which do not meet D . The resulting surface S'' has boundary $y_1 \cup y_2 \cup \dots \cup y_m \cup D$. Since the sum of the y_i 's represents 0 in $H_1(S; \mathbb{Z})$, the curves y_i cobound a collection of subsurfaces of S . Then we may glue a collection of parallel copies of these surfaces to the appropriate boundary components of S'' to form a new two-sided surface S''' with boundary D . Pushing the parallel copies of subsurfaces of S apart and away from S , we will find that S''' is embedded and disjoint from S . \square

Now returning to the proof of the theorem:

Applying the lemma to each of the link components L_i , we may find Seifert surfaces $\Sigma_1, \Sigma_2, \dots, \Sigma_n$ for L_1, L_2, \dots, L_n , respectively, such that $\Sigma \cap \Sigma_i = \emptyset$ for $i = 1, 2, \dots, n$. We show

Claim: (K, L, k, p, q, I) is a pairwise Dehn surgery description for (X, \tilde{X}) .

Proof of Claim: It is clear from the construction that (K, L, k, p, q, I) is a pairwise Dehn surgery description for some pair (X, X') with base space X . Let $\psi : X' \rightarrow X$ be the projection map. Then $\psi_*(\pi_1(X'))$ is the kernel of the homomorphism

$$\pi_1(X) \rightarrow H_1(X; \mathbb{Z}) \rightarrow \mathbb{Z}/k\mathbb{Z},$$

where the latter homomorphism is the projection

$$H_1(X; \mathbb{Z}) \rightarrow H_1(X; \mathbb{Z})/H = \mathbb{Z}/kp\mathbb{Z} \rightarrow \mathbb{Z}/k\mathbb{Z}.$$

But this kernel is precisely $\phi_*(\pi_1(\tilde{X}))$. Hence $X' = \tilde{X}$. \square

4 Casson-Walker invariants for pairs

In 1985, Andrew Casson defined an invariant λ for integral homology 3-spheres. Roughly, this invariant counts the signed equivalence classes of $SU(2)$ -representations of the fundamental group of the 3-manifold. This invariant was extended to an invariant for oriented rational homology 3-spheres by Kevin Walker in [W], and Christine Lescop derived a combinatorial formula extending the invariant to arbitrary closed, oriented 3-manifolds in [L].

A number of mathematicians have explored the Casson-Walker invariant for branched covers of links in S^3 . David Mullins computed the invariant for 2-fold branched covers in the case when the 2-fold branched cover is a rational homology sphere in [M]. More general results for $k > 2$ may be found in [GR]. The invariants for n -fold branched covers \overline{S}_K^3 of particular families of knots have been computed by J. Hoste, A. Davidow, and K. Ishibe in [H], [D], and [I]. Garoufalidis generalizes several of these formulas in [G].

We study the Casson-Walker invariants of pairs (X, \tilde{X}) . As in the early sections of the paper, we assume $\tilde{X} \rightarrow X$ is a torsion-split regular k -fold cyclic covering over a rational

homology sphere. In this section, we assume further that \tilde{X} is a rational homology sphere and that (X, \tilde{X}) has a Dehn surgery description (K, L, k, p, q, I) with \overline{S}_K^3 a rational homology sphere.

In what follows, for any pair of non-zero integers x and y which are relatively prime, let $s(x, y)$ denote the Dedekind sum defined by

$$s(x, y) = \text{sign}(y) \sum_{j=1}^{|x|} ((j/y))((jx/y))$$

where

$$((z)) = \begin{cases} 0 & z \in \mathbb{Z} \\ z - [z] - 1/2 & \text{else} \end{cases}$$

For a knot C in a rational homology sphere, let Δ_C denote the Alexander polynomial of C , normalized so that it is symmetric in $t^{1/2}$ and $t^{-1/2}$ and so that $\Delta_C(1) = 1$. We show

Theorem 4.1 *Let (K, L, k, p, q, I) be a pairwise Dehn surgery description for (X, \tilde{X}) . Then*

$$\lambda(\tilde{X}) = k\lambda(X) + q/p(\Delta_{\overline{K}}''(1) - \Delta_K''(1)) - ks(q, kp) + s(q, p) + \lambda(\overline{S}_K^3).$$

Here, \overline{S}_K^3 denotes the k -fold branched cyclic cover of S^3 branched along K and \overline{K} denotes the lift of K to \overline{S}_K^3 , as above.

Note that $\Delta_{\overline{K}}$ can be computed from Δ_K . This relationship is described in the appendix. We remark further that in the case $k = 2$, the invariant $\lambda(\overline{S}_K^3)$ can be computed whenever \tilde{X} is a rational homology sphere using the work of Mullins. For $k > 2$, the results of Garoufalidis and Rozansky apply. For certain families of knots, the invariant $\lambda(\overline{S}_K^3)$ can be computed for any value of k using the results of Hoste, Davidow, Ishibe, and Garoufalidis.

Proof of Theorem 4.1: Retain all notation from previous sections. We begin by noting that

$$\lambda(N) = (q/kp)\Delta_K''(1) + s(q, kp) \tag{1}$$

and

$$\lambda(\tilde{N}) = \lambda(\overline{S}_K^3) + (q/p)\Delta_{\overline{K}}''(1) + s(q, p) \tag{2}$$

by Proposition 6.2 of [W], since K is a knot in S^3 and since \overline{K} is homologous to 0 in \overline{S}_K^3 .

Now X is obtained from N by I -surgery on L . We know that $\lambda(X) - \lambda(N)$ may be obtained using the surgery formulae developed by Walker in [W]. These formulae depend on the coefficients I , as well as the link L and the Alexander polynomials of the components of L . Similarly \tilde{X} is obtained from \tilde{N} by I -surgery on each lift \tilde{L}_j of L in \tilde{N} .

Now by Proposition A.7 in the appendix, the Alexander polynomial of each component of \bar{L} is equal to that of the corresponding component of L . Therefore, since a neighborhood of each component $\bar{L}_{j,i}$ of \bar{L}_j in \tilde{N} is carried homeomorphically onto a neighborhood of L_j in N and the coefficients I correspond, it is clear that for any j , the Casson-Walker invariant of the manifold \tilde{N}_j obtained by doing I -surgery on \bar{L}_j in \tilde{N} obeys the formula $\lambda(\tilde{N}_j) - \lambda(\tilde{N}) = \lambda(X) - \lambda(N)$. But the lifts $\bar{\Sigma}_{j,i}$ of Σ_j are contained in disjoint copies of $S^3 - Y$ in \tilde{N} , as shown in the proof of Theorem 1.2, so $\bar{\Sigma}_{j,i} \cap \bar{\Sigma}_{l,m} = \emptyset$ if $i \neq m$. Then also $\lambda(\tilde{N}_{jk}) - \lambda(\tilde{N}_j) = \lambda(\tilde{N}_j) - \lambda(\tilde{N})$, where \tilde{N}_{jk} is the manifold obtained by doing I -surgery on the image of \bar{L}_k in \tilde{N}_j . It follows that

$$\lambda(\tilde{X}) - \lambda(\tilde{N}) = k(\lambda(X) - \lambda(N)). \quad (3)$$

The theorem follows after suitably combining equations (1) - (3). \square

We remark that analogous results for the generalizations of λ counting representations in $\mathrm{SO}(3)$, $\mathrm{U}(2)$, $\mathrm{Spin}(4)$, and $\mathrm{SO}(4)$ may be immediately obtained using the results of [C].

5 Cyclic and finite Dehn surgeries

In recent years, many new and exciting results have come to light concerning Dehn fillings of 3-manifolds with toral boundary. Among these are the Cyclic Surgery Theorem of Culler, Gordon, Luecke, and Shalen [CGLS] and the work of Boyer and Zhang concerning finite fillings [BZ1] and [BZ2].

We review some basic definitions.

Definition 5.1 *Let K be a knot homologous to 0 in an oriented 3-manifold Y . A **slope** of K is the unoriented isotopy class of a non-trivial simple closed curve in $\partial(Y - K)$. The **distance** between two slopes is their geometric intersection number.*

Recall that for any knot homologous to 0 in an oriented 3-manifold, the set of slopes of the knot is canonically isomorphic to $\overline{\mathbb{Q}}$. Thus, we denote by p/q the slope which is the isotopy class of a curve which is p times a meridian plus q times a longitude.

Definition 5.2 *With K and Y as above, suppose that p/q is a slope of K such that the manifold X which is the result of p/q Dehn surgery on K in Y has finite fundamental group. Then we call p/q a **finite surgery slope** of K . Similarly, if X has cyclic fundamental group, we call p/q a **cyclic surgery slope** of K .*

The results of Culler, Gordon, Luecke, and Shalen and of Boyer and Zhang state that for most irreducible knot complements $Y - K$ as above, there are at most 3 cyclic surgery slopes of K and at most 6 finite surgery slopes. Moreover these slopes may be distance at most 1 apart in the cyclic case and at most 5 apart in the finite case. A detailed survey of work in this area may be found in [B].

Here, we note that given a knot K homologous to 0 in a closed, oriented 3-manifold Y and a finite or cyclic surgery slope of K , our work leads to explicit examples of more such fillings. Specifically, let \overline{Y}_K denote the k -fold branched cyclic cover Y branched along K , and let \overline{K} denote the lift of K to \overline{Y}_K . We show

Theorem 5.3 *Let K be a knot homologous to 0 in a closed oriented 3-manifold Y . Let k , p , and q be integers with $k > 1$, $|p| \geq 1$, and $q \geq 1$. Then kp/q is a finite surgery slope of K if and only if p/q is a finite surgery slope of \overline{K} in \overline{Y}_K . Moreover if kp/q is a cyclic surgery slope of K , then p/q is a cyclic surgery slope of \overline{K} . Finally, if p/q is a cyclic surgery slope of \overline{K} and $p \neq 1$, then kp/q is a cyclic surgery slope of K .*

Proof: Let X denote the manifold resulting from kp/q -Dehn surgery on K , and let \tilde{X} denote the result of p/q -Dehn surgery on \overline{K} . As in the proof of Theorem 1.3, we may find a knot C and a link $L = (L_1, L_2, \dots, L_n)$ in S^3 satisfying

- there exist Seifert surfaces Σ and $\Sigma_1, \Sigma_2, \dots, \Sigma_n$ for C and L_1, L_2, \dots, L_n , respectively, with $\Sigma \cap \Sigma_j = \emptyset$ for $j = 1, 2, \dots, n$
- Y is the result of I -surgery on L for some $I = (i_1, i_2, \dots, i_n)$ with $i_j = \pm 1$
- K is the image of C in Y .

(Note that none of these steps requires Y to be a rational homology sphere.)

Now (C, L, k, p, q, I) is a Dehn surgery description for some pair (Z, \tilde{Z}) . But since (C, L_i) is a boundary link for $i = 1, 2, \dots, n$, it is clear that kp/q -surgery on C followed by I -surgery on the image of L yields the same manifold as I -surgery on L followed by kp/q -surgery on the image of C . Hence $X \cong Z$. Also, since (C, L_i) is a boundary link, we see that \overline{Y}_K may be obtained by I -surgery on the k lifts of L to \overline{S}_C^3 . Moreover p/q -surgery on \overline{C} in \overline{S}_C^3 followed by I -surgery on the images of each of the k lifts of L to \overline{S}_C^3 yields the same manifold as I -surgery on each of the lifts of L to \overline{S}_C^3 followed by p/q -surgery on the image \overline{K} of \overline{C} in \overline{Y}_K . Hence $\tilde{X} \cong \tilde{Z}$.

Now it follows from Theorem 1.2 that \tilde{X} is a regular k -fold cyclic covering space of X . Therefore the fundamental group of \tilde{X} is an index k subgroup of that of X . Then clearly $\pi_1(X)$ is finite if and only if $\pi_1(\tilde{X})$ is finite, and $\pi_1(X)$ is cyclic if $\pi_1(\tilde{X})$ is cyclic.

Finally, if $\pi_1(\tilde{X})$ is cyclic and $p \neq 1$, so $\pi_1(\tilde{X}) = \mathbb{Z}/p\mathbb{Z}$, then $\pi_1(X) = H_1(X) = \mathbb{Z}/kp\mathbb{Z}$, since $H_1(X; \mathbb{Z})$ has a $\mathbb{Z}/kp\mathbb{Z}$ summand and $\pi_1(\tilde{X})$ is index k in $\pi_1(X)$. This proves the theorem. \square

In light of [CGLS], [BZ1], and [BZ2], then, finding a finite or cyclic surgery slope p/q for some knot K homologous to 0 in an oriented 3-manifold Y severely restricts the set of possible finite and cyclic surgery slopes not only of K itself, but also of lifts of K to branched covers of Y branched along K of all orders dividing p . For example, it is a result of Fintushel and Stern [FS] that 18- and 19-surgeries on the $(-2, 3, 7)$ pretzel knot yield lens spaces. It follows that 1-surgery on the lift of the $(-2, 3, 7)$ pretzel knot to either the 18- or 19-fold branched cover of S^3 branched along the pretzel knot yields S^3 . Further, we find that p -surgery on the lift of the knot to the $(18/p)$ -fold branched cover of S^3 branched along the knot also yields a lens space for $p = 2, 3, 6$, or 9 .

Finally, we note that the results of [CGLS], [BZ1], and [BZ2] may be strengthened for manifolds of the form $\overline{Y}_K - \overline{K}$. We have the following corollary, where Y , K , \overline{Y}_K , and \overline{K} are defined as above:

Corollary 5.4 *Suppose $Y - K$ is irreducible and is not a Seifert fibered space. Then \overline{K} has at most one cyclic surgery slope p/q with $p \neq 1$.*

If also $Y - K$ is not a cable on the twisted I -bundle over the Klein bottle, then the distance between any two finite surgery slopes of \overline{K} is at most $5/k$.

If $Y - K$ is also hyperbolic, then the distance between a cyclic surgery slope p/q with $p \neq 1$ and any finite surgery slope is at most $2/k$. Moreover the distance between any two finite surgery slopes is at most $3/k$.

Proof: By Theorem 5.3, for any p and q with $|p| \geq 1$ and $q \geq 1$, we know that p/q is a finite surgery slope of \overline{K} if and only if kp/q is a finite surgery slope of K and that kp/q is a cyclic surgery slope of K if p/q is a cyclic surgery slope of \overline{K} with $p \neq 1$. The distance between kp'/q' and kp/q is k times the distance between p'/q' and p/q . The assertions follow from the restrictions on the distance between cyclic (resp. finite) surgery slopes kp/q for K imposed by [CGLS], [BZ1], and [BZ2]. \square

A Alexander polynomials in \overline{S}_K^3

In this section, we relate the Alexander polynomials $\Delta_{\overline{K}}$ and $\Delta_{L_{j,i}}$ in \overline{S}_K^3 to Δ_K and Δ_{L_j} , respectively. These results are used in Section 4.

We begin by recalling the definition of the Alexander polynomial of a knot in a rational homology sphere. Details can be found in [M].

Let C be a knot in a rational homology sphere, and denote by Z the complement of a tubular neighborhood of C . Further, denote by \tilde{Z} the infinite cyclic cover of Z determined by $\pi_1(Z) \rightarrow H_1(Z; \mathbb{Z})/\text{torsion}$. Then $H_1(\tilde{Z}; \mathbb{Q})$ is a module over $\mathbb{Q}[\mathbb{Z}]$.

Now $\mathbb{Q}[\mathbb{Z}]$ is a principal ideal domain, so

$$H_1(\tilde{Z}; \mathbb{Q}) \cong \mathbb{Q}[\mathbb{Z}]/(p_1) \oplus \mathbb{Q}[\mathbb{Z}]/(p_2) \oplus \dots \oplus \mathbb{Q}[\mathbb{Z}]/(p_r).$$

We define the *order* of $H_1(\tilde{Z}; \mathbb{Q})$ to be the product ideal $(p_1 p_2 \dots p_r)$.

Definition A.1 *An Alexander polynomial $\Delta_C(t)$ of C is any polynomial generating the order of $H_1(\tilde{Z}; \mathbb{Q})$. We also call $\Delta_C(t)$ an **Alexander polynomial** of the $\mathbb{Q}[\mathbb{Z}]$ -module $H_1(\tilde{Z}; \mathbb{Q})$.*

This is well-defined up to multiplication by polynomials of the form ct^k , where $c \in \mathbb{Q}$, where t generates the deck transformations of the covering $\tilde{Z} \rightarrow Z$, and where $k \in \mathbb{Z}$. Henceforth we write $p(t) \equiv q(t)$ if $p(t)$ and $q(t)$ are polynomials with $p(t) = ct^k q(t)$ for some $c \in \mathbb{Q}$ and some $k \in \mathbb{Z}$.

We first relate $\Delta_{\bar{K}}$ to Δ_K . We prove the following theorem, which was pointed out to me by Steve Boyer. The proof offered is that of Boyer.

Proposition A.2 *(Boyer)*

$$\Delta_{\bar{K}}(t^k) \equiv \prod_{j=0}^{k-1} \Delta_K(\zeta^j t)$$

where ζ is a primitive k th root of unity.

Proof (Boyer): Let A be the matrix of multiplication by t in $H_1(\tilde{Z}; \mathbb{Q})$ with respect to the \mathbb{Q} -vector space structure. We first show

Lemma A.3 *(Boyer)* $\Delta_K(t) \equiv |A - tI|$

Proof of lemma (Boyer): Since $H_1(\tilde{Z}; \mathbb{Q})$ decomposes as

$$\mathbb{Q}[t, t^{-1}]/p_1(t) \oplus \mathbb{Q}[t, t^{-1}]/p_2(t) \oplus \dots \oplus \mathbb{Q}[t, t^{-1}]/p_r(t),$$

we see that $A = \oplus_{j=1}^r A_j$, where A_j is the matrix of multiplication by t in $\mathbb{Q}[t, t^{-1}]/p_j(t)$. Thus it suffices to show that

$$|A_j - tI| \equiv p_j(t).$$

But writing $p_j(t) = b_0 + b_1t + b_2t^2 + \dots + b_{s-1}t^{s-1} + t^s$, we see that $\mathbb{Q}[t, t^{-1}]/p_j(t)$ has basis $1, t, t^2, \dots, t^{s-1}$. Then

$$A_j = \begin{bmatrix} 0 & 0 & \dots & 0 & -b_0 \\ 1 & 0 & \dots & 0 & -b_1 \\ 0 & 1 & \dots & 0 & -b_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -b_{s-1} \end{bmatrix}$$

It follows that $|A_j - tI| = (-1)^s p_j(t)$. \square

Now define a new $\mathbb{Q}[t, t^{-1}]$ -module structure on $H_1(\tilde{Z}; \mathbb{Q})$ with multiplication $t * m = t^k m$. Clearly $H_1(\tilde{Z}; \mathbb{Q})$ with the new multiplication is again a finitely generated, torsion $\mathbb{Q}[t, t^{-1}]$ -module and hence a finite dimensional \mathbb{Q} -vector space. We show

Lemma A.4 (Boyer) *Let $\Delta_k(t)$ denote an Alexander polynomial for $H_1(\tilde{Z}; \mathbb{Q})$ with the new $\mathbb{Q}[t, t^{-1}]$ -module structure. Then*

$$\Delta_k(t^k) \equiv \prod_{j=0}^{k-1} \Delta(\zeta^j t)$$

Proof of lemma (Boyer): If A is the matrix of multiplication by t in $H_1(\tilde{Z}; \mathbb{Q})$ with the original $\mathbb{Q}[t, t^{-1}]$ -module structure, then A^k is the matrix of multiplication by t in $H_1(\tilde{Z}; \mathbb{Q})$ with the new $\mathbb{Q}[t, t^{-1}]$ -module structure. Therefore by the previous lemma,

$$\Delta_k(t^k) \equiv |A^k - t^k I| \tag{4}$$

$$= \prod_{j=0}^{k-1} |A - \zeta^j t I| \tag{5}$$

$$= \prod_{j=0}^{k-1} \Delta(\zeta^j t) \tag{6}$$

This proves the lemma. \square

But now note that an Alexander polynomial $\Delta_{\overline{K}}(t)$ is an Alexander polynomial of $H_1(\tilde{Z}; \mathbb{Q})$ with the second $\mathbb{Q}[t, t^{-1}]$ -module structure, while $\Delta_K(t)$ is an Alexander polynomial of $H_1(\tilde{Z}; \mathbb{Q})$ with the original $\mathbb{Q}[t, t^{-1}]$ -module structure. The theorem follows. \square

Thus, renaming $u = t^n$, say, we obtain $\Delta_{\overline{K}}(u)$ in terms of Δ_K .

This can be further simplified as follows: write

$$\Delta_K(t) = c_0 + c_1(t + t^{-1}) + c_2(t^2 + t^{-2}) + \dots + c_n(t^n + t^{-n}). \quad (7)$$

Applying Boyer's theorem to (7) and noting that all cross-terms cancel, we see that

$$\Delta_{\overline{K}}(t^k) \equiv c_0^k + c_1^k \prod_{j=0}^{k-1} (\zeta^j t + \zeta^{-j} t^{-1}) + c_2^k \prod_{j=0}^{k-1} (\zeta^{2j} t^2 + \zeta^{-2j} t^{-2}) + \dots + c_n^k \prod_{j=0}^{k-1} (\zeta^{nj} t^n + \zeta^{-nj} t^{-n}). \quad (8)$$

In particular, if k is odd, note that $\prod_{j=0}^{k-1} (\zeta^{rj} t^r + \zeta^{-rj} t^{-r}) = t^{rk} + t^{-rk}$ for any r . Then equation (8) becomes

$$\Delta_{\overline{K}}(t^k) \equiv c_0^k + c_1^k(t^k + t^{-k}) + c_2^k(t^{2k} + t^{-2k}) + \dots + c_n^k(t^{nk} + t^{-nk}) \quad (9)$$

and finally

$$\Delta_{\overline{K}}(u) \equiv c_0^k + c_1^k(u + u^{-1}) + c_2^k(u^2 + u^{-2}) + \dots + c_n^k(u^n + u^{-n}). \quad (10)$$

We now turn to relating $\Delta_{\overline{L}_{j,i}}$ to Δ_{L_j} . We return to the convention that Δ_C is symmetric in $t^{1/2}$ and $t^{-1/2}$ and that $\Delta_C(1) = 1$, so that Δ_C is uniquely defined.

For the remainder of the paper, for any rational homology sphere M , let $lk_M(-, -)$ denote the linking number in M .

Lemma A.5 *Given knots C and C' in a rational homology sphere M such that $lk_M(C, C') = 0$. Let \hat{M} denote the manifold resulting from p/q -Dehn surgery on C' in M . Then $lk_{\hat{M}}(\hat{C}, \hat{D}) = lk_M(C, D)$ for any knot D in M , where \hat{C} and \hat{D} are the images of C and D in \hat{M} .*

Proof: Fix a knot D in M . Suppose C represents an element of order m in $H_1(M; \mathbb{Z})$. Since $lk_M(C, C') = 0$, we may choose a two-sided surface Σ with boundary m times C which is disjoint from a neighborhood of C' and which meets D transversely. Then Dehn surgery on C' does not affect a neighborhood of Σ and hence does not affect the oriented intersection of Σ and D . The assertion follows. \square

We now show

Lemma A.6 *Let C and D be knots in a rational homology sphere M . Suppose C is homologous to 0 and bounds a Seifert surface Σ disjoint from D satisfying $lk_M(D, \alpha_j) = 0$ for $j = 1, 2, \dots, 2g$, where $\alpha_1, \alpha_2, \dots, \alpha_{2g}$ is a collection of simple closed curves on Σ representing a basis of $H_1(\Sigma; \mathbb{Z})$. Let M' denote the manifold resulting from p/q -Dehn surgery on D in M , and let C' denote the image of C in M' . Then the Alexander polynomial of C' in M' is equal to the Alexander polynomial of C in M .*

Proof: Let Σ' be the image of Σ in M' . Let $\alpha'_1, \alpha'_2, \dots, \alpha'_{2g}$ denote the images of $\alpha_1, \alpha_2, \dots, \alpha_{2g}$, respectively, in M' . The Alexander polynomial Δ_C of C in M is the determinant of the matrix A with entries $a_{i,j} = lk_M(\alpha_i^+, \alpha_j) - tlk_M(\alpha_i^-, \alpha_j)$, while the Alexander polynomial $\Delta_{C'}$ of C' in M' is the determinant of the matrix A' with entries $a'_{i,j} = lk_{M'}(\alpha_i'^+, \alpha'_j) - tlk_{M'}(\alpha_i'^-, \alpha'_j)$. (Here x^+ and x^- denote the plus- and minus-pushoffs of a simple closed curve x on Σ or Σ' .) (See [W, Appendix B], for example.)

But since $lk_M(D, \alpha_j) = 0$ for $j = 1, 2, \dots, 2g$, we may apply Lemma A.5 to show that $lk_{M'}(\alpha_i'^+, \alpha'_j) = lk_M(\alpha_i^+, \alpha_j)$ and $lk_{M'}(\alpha_i'^-, \alpha'_j) = lk_M(\alpha_i^-, \alpha_j)$ for all i and j . It follows that $A' = A$, and hence $\Delta_{C'} = \Delta_C$. \square

Proposition A.7 *Let (K, L) be a boundary link in S^3 , and let \bar{L} be a lift of L in the branched k -fold cover \bar{S}_K^3 for some k . The Alexander polynomial of \bar{L} in \bar{S}_K^3 is equal to the Alexander polynomial of L in S^3 .*

Proof: Choose a link $C = (C_1, C_2, \dots, C_n)$ in S^3 and surgery coefficients $r = (r_1, r_2, \dots, r_n)$ so that r -Dehn surgery on C yields S^3 and so that the image K' of K is an unknot. (See [R, Section 6D], for example.) Isotoping C as necessary, changing crossings of C and L but changing no crossings of $C \cup K$, we may assume that L , K , and C_i bound Seifert surfaces Σ_L , Σ_K , and Σ_i , respectively, with $\Sigma_L \cap \Sigma_K = \Sigma_L \cap \Sigma_i = \emptyset$ for $i = 1, 2, \dots, n$.

Let L' denote the image of L under r -surgery on C . Let $\alpha_1, \alpha_2, \dots, \alpha_{2g}$ be a collection of simple closed curves on Σ representing a basis of $H_1(\Sigma; \mathbb{Z})$. By Lemma A.5, the linking number of the image of any component C_i with the image of any curve α_j is 0 at any stage of the sequence of surgeries on C_1, C_2, \dots, C_n . Then we may apply Lemma A.6 repeatedly to show that the Alexander polynomial $\Delta_{L'} = \Delta_L$. Furthermore, the linking number of K' and the image of any α_j in S^3 at the end of the surgery sequence is 0. Then applying Lemma 3.2, we find that (L', K') and (L', C'_i) are boundary links, where the C'_i are the images of the C_i in S^3 after the surgery. In fact, since K' is an unknot, we see that (L', K') is splittable.

Now consider the k -fold branched cover of S^3 branched along the unknot K' , which is again S^3 . Since (L', K') is splittable, we see that L' has k lifts $\bar{L}'_1, \bar{L}'_2, \dots, \bar{L}'_k$ lying in disjoint 3-balls, and the Alexander polynomials satisfy $\Delta_{\bar{L}'_i} = \Delta_{L'}$ for $i = 1, 2, \dots, k$. Let \bar{C}' denote the inverse image of C' in $\bar{S}_{K'}^3$.

Finally, it is clear that surgery on \bar{C}' with appropriate coefficients yields \bar{S}_K^3 and takes \bar{L}'_i onto \bar{L}_i for $i = 1, 2, \dots, k$. The assertion follows. \square

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